December 13, 2018, 2 hours

Solve only 10 Problems out of 13

1. Given the vector $\overrightarrow{V} = r^2\left(\cos\theta\hat{r} + \cos\phi\hat{\theta} - \cos\theta\sin\phi\hat{\phi}\right)$. Find $\overrightarrow{\nabla}\cdot\overrightarrow{V}$ and evaluate the volume integral $\int_V \overrightarrow{\nabla}\cdot\overrightarrow{V}dv$ where V is the volume of the upper hemisphere of radius R centered at the origin. Check the divergence theorem by comparing with the surface integral $\oint_S \overrightarrow{V}\cdot d\overrightarrow{a}$ where S is the boundary of V consisting of the upper hemisphere and the flat disk of radius R in xy plane. Hint: To find $\overrightarrow{V}\cdot d\overrightarrow{a}$ on flat disk use $\hat{r} = \cos\theta\,\overrightarrow{k} + \sin\theta\left(\cos\phi\,\overrightarrow{i} + \sin\phi\,\overrightarrow{j}\right)$, $\hat{\theta} = -\sin\theta\,\overrightarrow{k} + \cos\theta\left(\cos\phi\,\overrightarrow{i} + \sin\phi\,\overrightarrow{j}\right)$, $\hat{\phi} = -\sin\phi\,\overrightarrow{i} + \cos\phi\,\overrightarrow{j}$.

2. Consider the vector field $\overrightarrow{F} = (x+y)\,\overrightarrow{i} + (x-y)\,\overrightarrow{j}$. Evaluate the line integral $\oint_C \overrightarrow{F}.d\overrightarrow{r}$ where C is the square of vertices $O=(0,0)$, $A=(1,-1)$, $B=(2,0)$, $C=(1,1)$. Is the vector $\overrightarrow{F}$ conservative. Hint: Find equations of the lines OA, AB, BC, CA.

3. Let $\{\overrightarrow{e}_1, \overrightarrow{e}_2, \overrightarrow{e}_3\}$ be a basis of a three dimensional vector space V_3 and T an operator that transforms this basis to the vectors $\{\overrightarrow{e}'_1, \overrightarrow{e}'_2, \overrightarrow{e}'_3\}$ where $\overrightarrow{e}'_1 = T(\overrightarrow{e}_1) = \overrightarrow{e}_1 + \overrightarrow{e}_3$, $\overrightarrow{e}'_2 = T(\overrightarrow{e}_2) = 2\overrightarrow{e}_1 + \overrightarrow{e}_2$, $\overrightarrow{e}'_3 = T(\overrightarrow{e}_3) = 3\overrightarrow{e}_2 + \overrightarrow{e}_3$. Show that the matrix representation of the operator T where $T\overrightarrow{e}_i = \sum_{j=1}^{3} T_{ji}\overrightarrow{e}_j$ is $T = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 1 & 0 & 1 \end{pmatrix}$. Is T invertible? Find the inverse of T by first finding the adjoint elements.

4. Find the eigenvalues and *normalized* eigenvectors of the matrix

$$A = \begin{pmatrix} \cos\theta & \sin\theta e^{-i\varphi} \\ \sin\theta e^{i\varphi} & -\cos\theta \end{pmatrix}$$

Express your answer in terms of $\sin\frac{\theta}{2}$ and $\cos\frac{\theta}{2}$.

5. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an orthogonal matrix satisfying $A^TA = I$ where I is the identity matrix. Find the most general solution for a, b, c, d satisfying the orthogonality condition. Assume $\det A = 1$ then express your answer in terms of trigonometric functions. Hint: Let $a = \cos\theta$, $b = \cos\theta'$.

6. Given the function $f(x) = \{a - |x|, -a \le x < a\}$ of period $2a$. Is $f(x)$ even or odd? Find the Fourier series for $f(x)$. Use the series to show that $\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$.

7. Given the function $f(x) = \begin{cases} 1-|x|, & -a \le x \le a \\ 0, & \text{otherwise} \end{cases}$. Is the function $f(x)$ even or odd? Find the Fourier transform $F(\omega)$ of $f(x)$. Is $F(\omega)$ even or odd.

8. Verify the delta function expansion

$$\delta(1+x) = \sum_{n=0}^{\infty}(-1)^n\frac{2n+1}{2}P_n(x)$$

Hint: Use the generating function $g(x,t) = \left(1 - 2xt + t^2\right)^{-\frac{1}{2}}$ to find the value of $P_n(-1)$.

9. Use the recursion relation for Hermite polynomials

$$2xH_n(x) = H_{n+1}(x) + 2nH_{n-1}(x)$$

and orthogonality condition $\int\limits_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\, dx = \delta_{nm}\sqrt{\pi}2^n n!$ to evaluate the integral

$$\int\limits_{-\infty}^{\infty} x^2 e^{-x^2} H_n(x) H_n(x)\, dx = \sqrt{\pi}2^n n! \left(n + \frac{1}{2}\right)$$

10. Find the analytic function $f(z) = u(x,y) + iv(x,y)$ where $u(x,y) = x^3 - 3xy^2$.

11. Derive the Laurent series for the function $f(z) = \frac{1}{z^2(z+2)}$ around the point $z_0 = -2$ for $|z+2| < 2$. Verify the values of a_0 and a_{-1} in the expansion $f(z) = \sum\limits_{n=-\infty}^{n=\infty} a_n (z - z_0)^n$ by comparing with the formula $a_n = \frac{1}{2\pi i} \oint dz \frac{f(z)}{(z-z_0)^{n+1}}$. Hint: Write $z = (z+2) - 2 = -2\left(1 - \frac{z+2}{2}\right)$ and to find the series $\frac{1}{(1-x)^2}$ differentiate the series $\frac{1}{1-x}$.

12. Prove that

$$\int\limits_0^{2\pi} \frac{d\theta}{1 - 2t\cos\theta + t^2} = \frac{2\pi}{1 - t^2}, \qquad 0 < t < 1$$

by evaluating an appropriate contour integral. Hint: let C be the unit circle and $z = e^{i\theta}$.

13. Find the residues of the function

$$f(z) = \frac{e^z}{z^2(z^2+9)}$$

about the isolated singular points. Use theorem of residues to deduce the value of the integral $\oint\limits_C dz f(z)$ where C is the circle of center at the origin and radius equal to 4.